FEM Guideline

Based on Numerical Methods for PDE taught by Prof. Ralf Hiptmair at ETH Zurich Spring semester 2012

# Model equation

## Variational form aka weak form

Recipe: Multiply the PDE with an abstract test function and integrate it over the whole domain

Apply Green’s first theorem to the LHS

By choosing on , the second term vanishes. We are perfectly allowed to define on-the-fly, since we haven’t defined it more specifically up to now.

This is called the variational or weak form of the PDE.

**Sidenote: Dirichlet boundary conditions**

Choosing on the boundary is not as arbitrary as it might seem. Rather it means that Dirichlet boundary conditions apply, i.e. the values on the boundary are prescribed (pinning conditions). If that were not the case, i.e. on the boundary, then (homogeneous) Neumann boundary condition apply (free boundary).

Dirichlet boundary conditions are imposed on the basis functions, unlike Neumann boundary conditions.

**Sidenote: Neumann boundary conditions**

If Dirichlet BC do not apply, i.e. if on then the second term does not vanish. To get rid of it anyway, must be zero, implying the so called homogeneous Neumann boundary condition:

If non-homogeneous Neumann BC are given explicitly, i.e.

Then the variational form reads

You may wonder how Dirichlet boundary conditions are incorporated. Dirichlet boundary conditions are handled later with the basis functions. Unlike Neumann boundary conditions, Dirichlet boundary conditions are not

# Discretisation: Basis functions

First of all, the domain is discretised to an unstructured mesh. An unstructured mesh is defined by its nodes and its edges. A common mesh is a triangulation, i.e. the domain consists of small triangles.

## Ritz-Galerkin discretisation

is the offset function holding boundary values, is the vector holding result coefficients for interior nodes and denotes a basis function, where is the number of nodes in the mesh. is an instance of this basis function at node .

**Sidenote: Offset functions**

In the above discretisation, is called a “offset function”. Offset functions enable the incorporation of Dirichlet BC. In fact, is a vector which is nonzero only for the boundary nodes. For every boundary node, it holds its prescribed boundary value. Although the index 0 might be misleading, is not an initial condition here.

## Basis functions

Technically, a basis consists of several local shape functions, i.e. . There is a local shape function for every node. Therefore, every triangle supports three shape functions, while all other shape functions are zero on this triangle. Hence, we can just look at one triangle to understand all triangles (aka the whole domain). NB: If the finite elements were quadrilaterals, then every quadrilateral would support four shape functions.

Every basis function (aka shape functions) must satisfy certain criteria.

* should be at least once differentiable

There is a variety of basis functions available, the easiest being linear basis functions.

Tools: Linear basis functions, barycentric coordinates

Expert tool: Polynomial basis functions

Generalisation: Quadrilaterals, polygons

# Solving variational form 🡪 Matrix equations

The variational form can be solved by inserting the discrete basis into it.

Now discretise in the same fashion as except that we don’t care about boundary terms, i.e.

## Computation of Galerkin matrix (LHS)

Incorporation of Dirichlet BC On the fly introduction: Bilinear form

Insight: Sparsity of Galerkin matrix. Nonzero entries only along edges

Computation of Local stiffness matrices (vertex-wise)  
Dimension depends on whether FE is triangle, quadrilateral etc.

Tool: Local indexing to Global indexing

Tool: Quadrature rules

Tool: Transformation rules for integral and gradient

Subtool: Boundary approximation

## Computation of RHS

Computation of element load vector

Boundary conditions (Dirichlet: FreeDofs trick, Neumann: Alteration of Galerkin matrix)